CHAPTER 2

Laplace Transform, Transfer Function and Electrical Systems Modeling

By
Dr. Ayman Yousef

Modified By
Dr. Emad Sami

Differential Equations

Differential Equation means that:

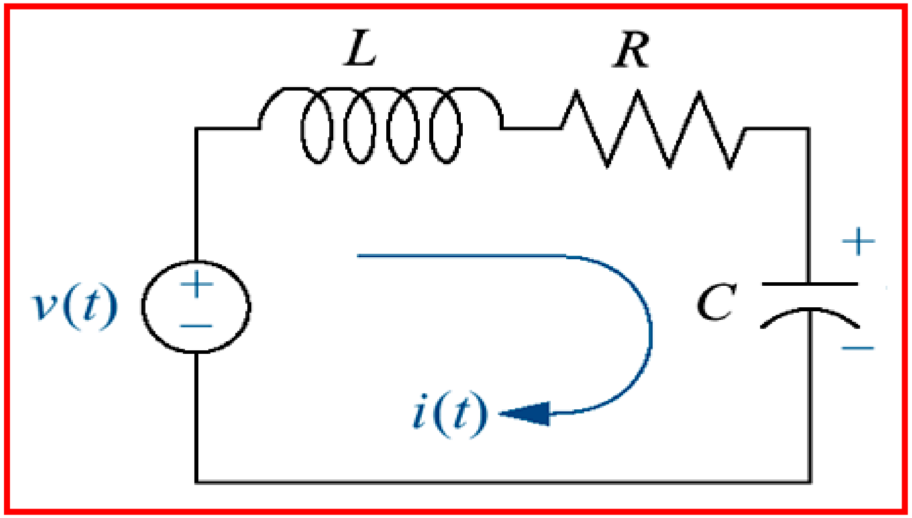
states how a rate of change "differential" in one variable is related to an other variable.

are equations generally contains derivatives and integrals of a dependent variable with respect to an independent variables.

Example For Differential Equation:

As shown in the figure: Series *RLC* circuit has applied voltage *v* :

$$v(t) = Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int i(t)dt$$



$v(t), R, L, C$ Independent or Known variables.

$i(t)$ Dependent or Unknown variable and to be known its value you must solve this equation.

How Can I Solve Differential Equation?

Solution Of Differential Equation Can Be Done Using Either of:

Laplace Transform Table.

Partial Fraction Expansion Technique.

Laplace Transform

Laplace Transform Method is famous method used in solving the differential equations because:

Both transient component and steady-state component of the solution can be obtained simultaneously.

a function of time

↓

Laplace Transform

↓

a function of frequency

Difference between t-domain and s-domain

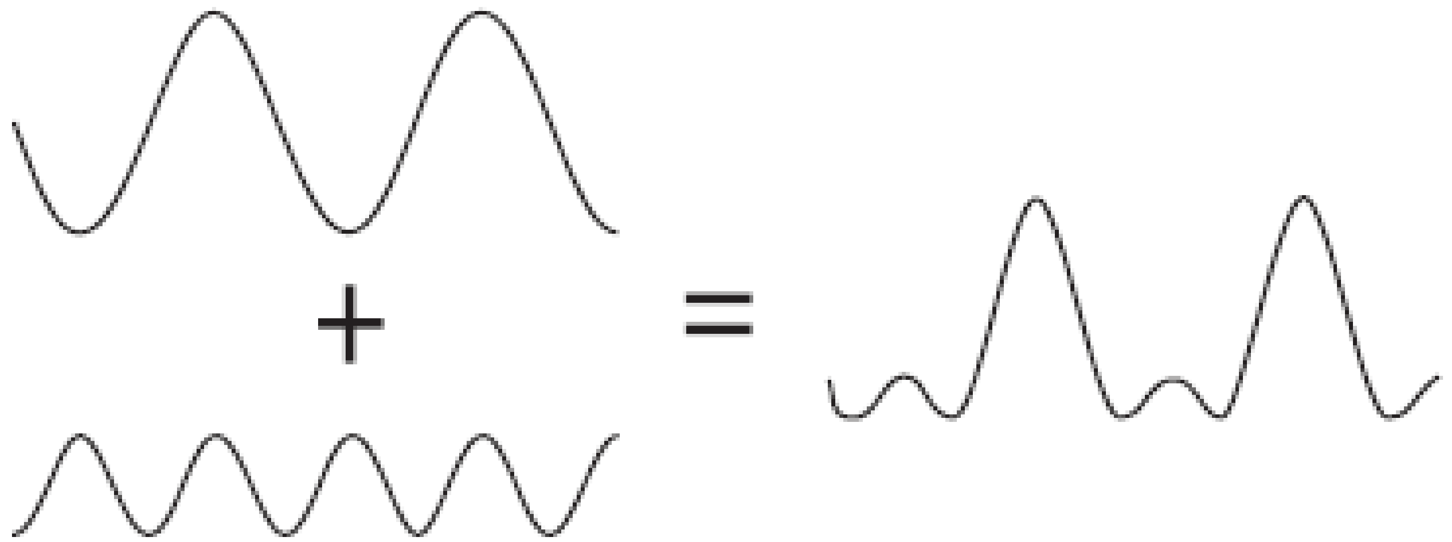
t-domain and s-domain:

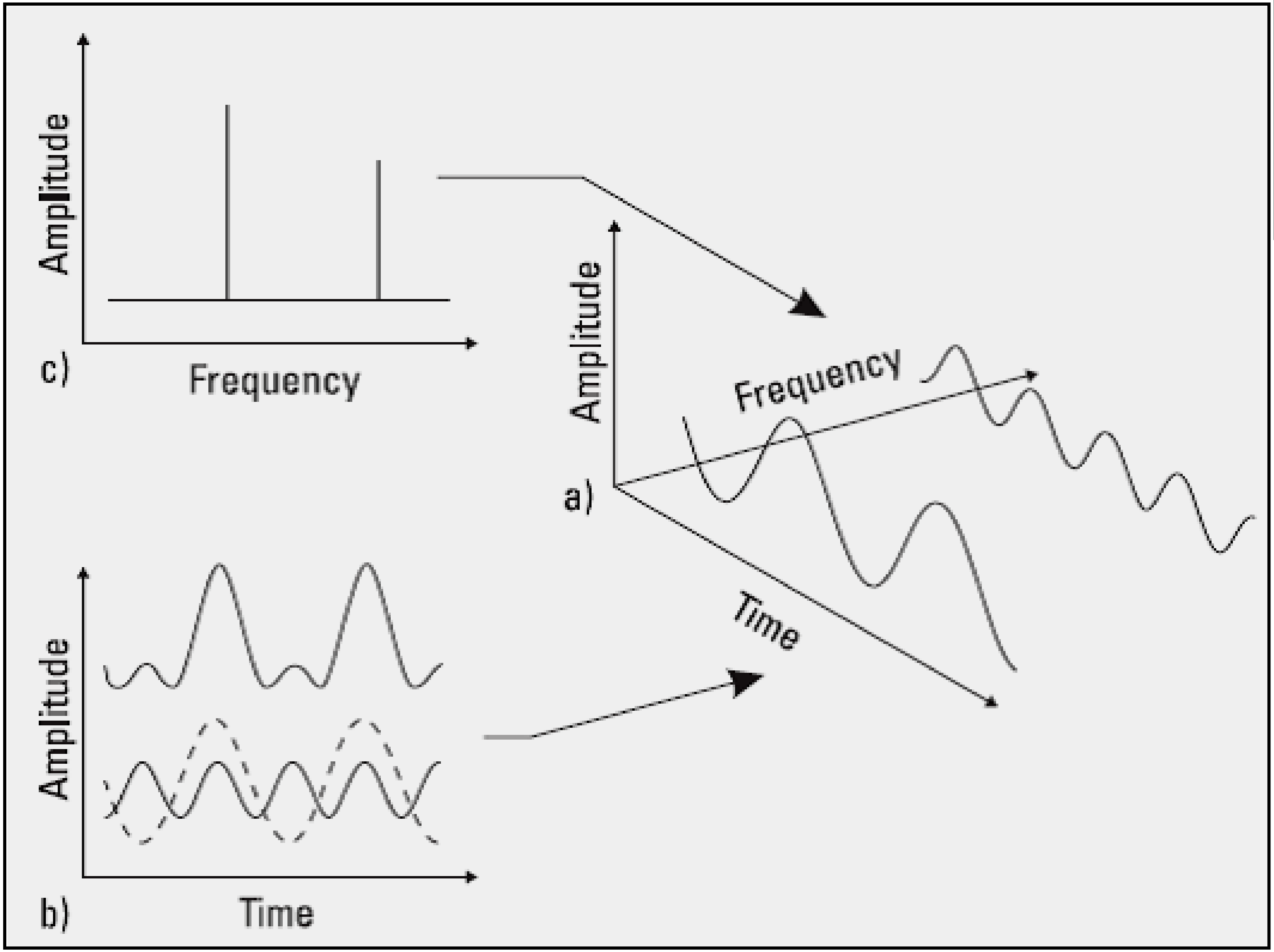
*** These are two ways of viewing the same thing.**

*** Are used in understanding the behavior of a dynamic system (dynamic system: is the system that has variables change w.r.t time).**

Time domain: t-domain	Freq. domain: s-domain
* we measure how long something takes.	we measure how fast or slow it is.
* what happens in the time domain as temporal	* what happens in the time domain as spectral

- Any real-world signal can be represented by only one combination of sine waves.
- In the diagram, a waveform is represented as the sum of two sine waves.



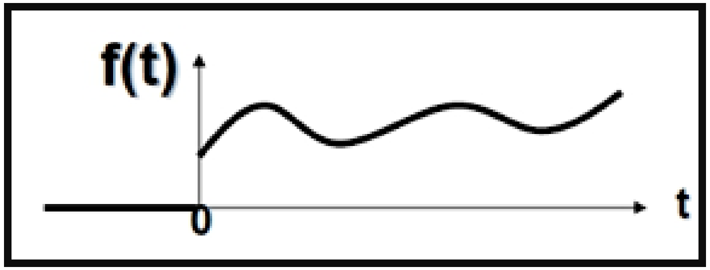

Amplitude
c)
Frequency
Amplitude
a)
Frequency
Time
Amplitude
b)
Time

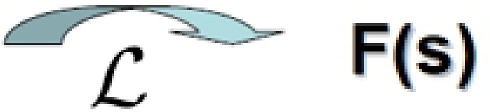
Laplace transform

- Definition: For a function f(t) (f(t)=0 for t<0),

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

(s: complex variable)





- We denote Laplace transform of f(t) by F(s).

Properties of Laplace transform

1- Differentiation

the Laplace transform of a time derivative is

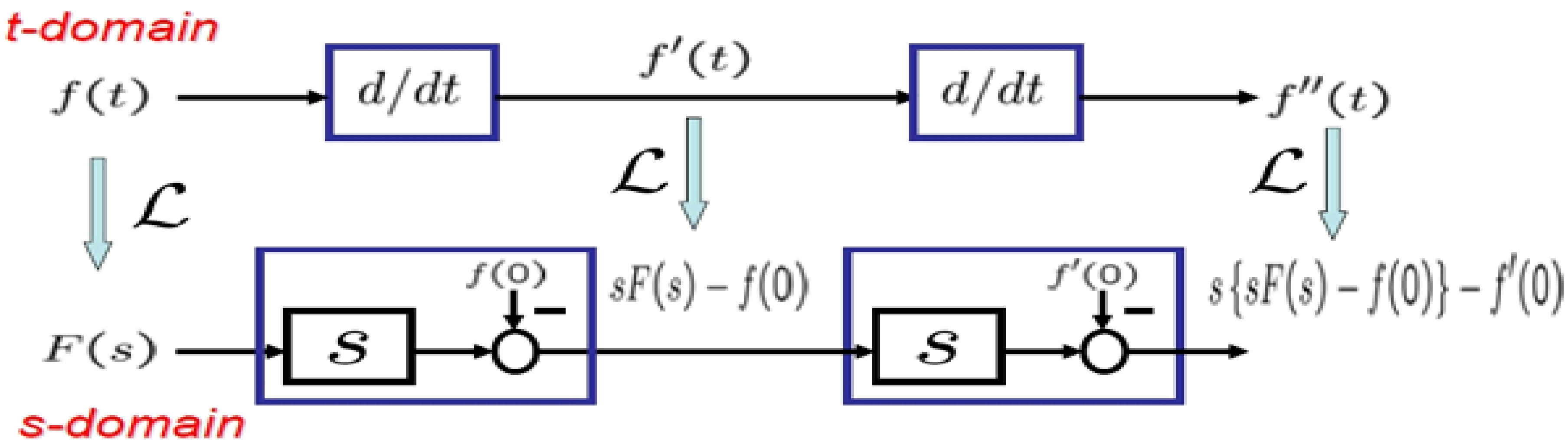
$$\mathcal{L}[f(t)] = F(s)$$

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots\ldots - f^{(n-1)}(0)$$

where $f(0), f'(0)$ are the initial conditions, or the values of $f(t)$, $d/dt\, f(t)$ etc. at $t = 0$

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$



2- Linearity:

$$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

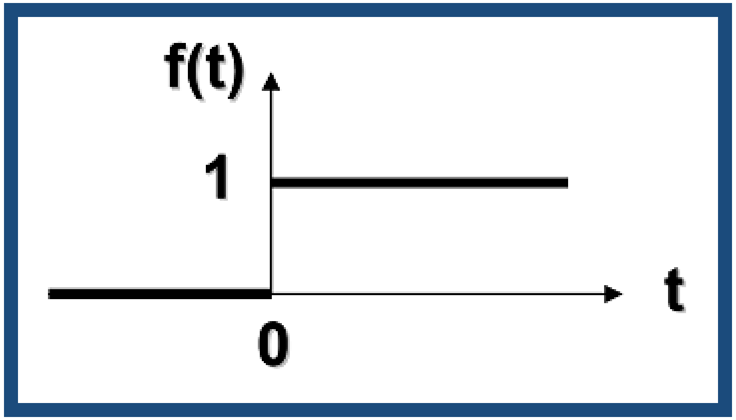
Unit step function

$$\mathcal{L}[1] = \int_0^\infty e^{-st}dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{0}{-s} - \frac{1}{-s}$$

$$= \frac{1}{s}$$

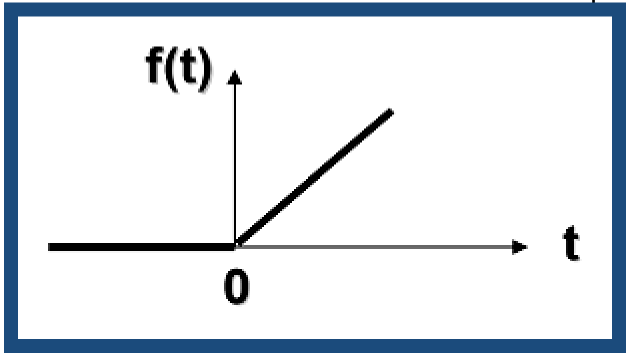


$$f(t) = u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[1] = \frac{1}{s}$$

Unit ramp

$$f(t) = u(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\mathcal{L}[t] = \int_0^\infty e^{-st} t\, dt = \left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{-s} \right]_0^\infty = \frac{1}{s^2}$$

similarly

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

Exponential function

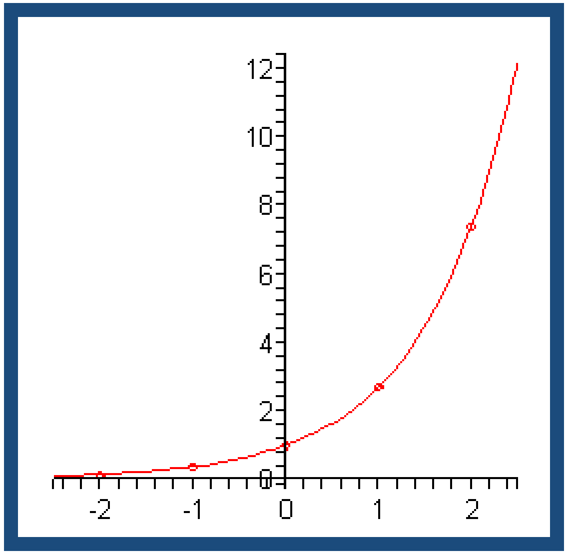
$$f(t) = e^{at}$$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at} e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{0}{-(s-a)} - \frac{1}{-(s-a)}$$

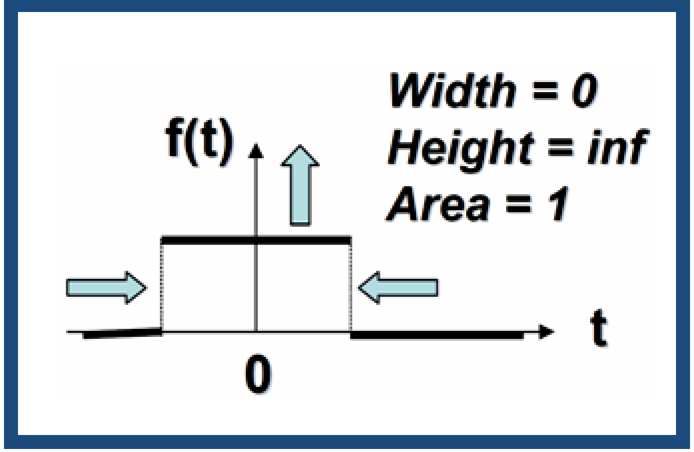
$$= \frac{1}{s-a}$$



$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

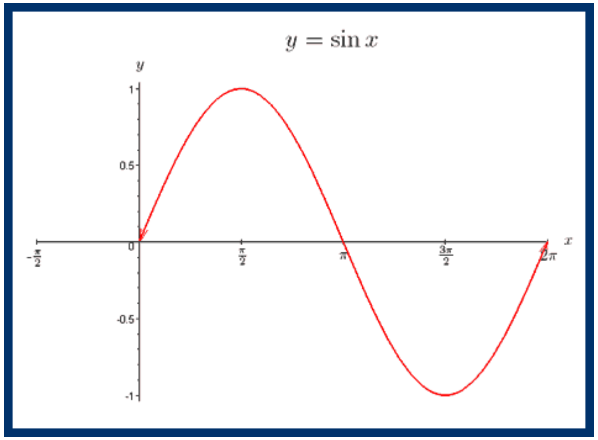
Impulse function

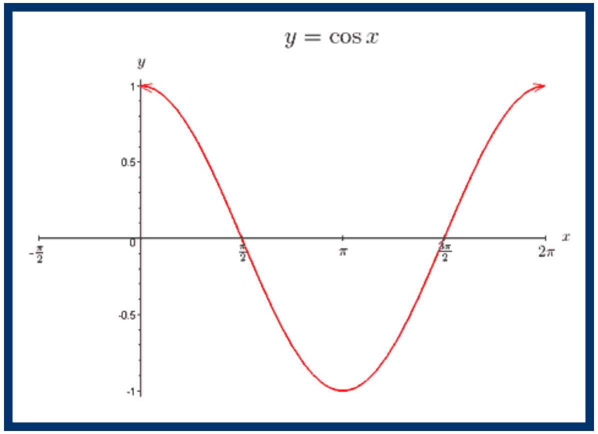
$$f(t) = \delta$$



$$\mathcal{L}[\delta(t)] = \int_{0-}^{\infty} \delta(t) e^{-st} dt = e^{-st}\Big|_{t=0} = 1$$

Sine and cosine functions





$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}$$

3- Integration:

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$$

4- Constant Multiplication:

$$\mathscr{L}[af(t)] = aF(s)$$

5- Real Shift Theorem:

$$\mathscr{L}[f(t-T)] = \mathrm{e}^{-Ts}F(s) \quad \text{for} \quad T \geq 0$$

6- Initial Value Theorem:

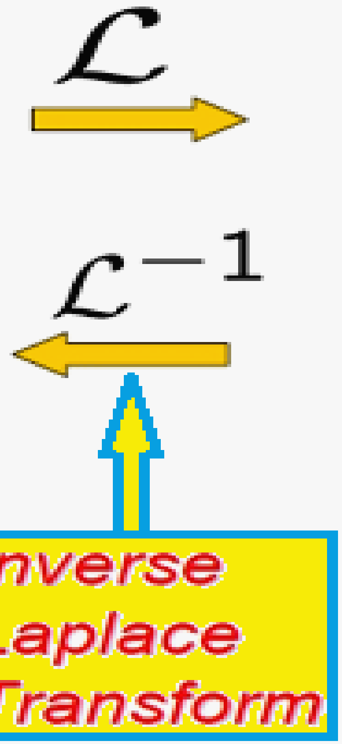
$$f(0) = \lim_{t \to 0} [f(t)] = \lim_{s \to \infty} [sF(s)]$$

7- Final Value Theorem:

$$f(\infty) = \lim_{t \to \infty} [f(t)] = \lim_{s \to 0} [sF(s)]$$

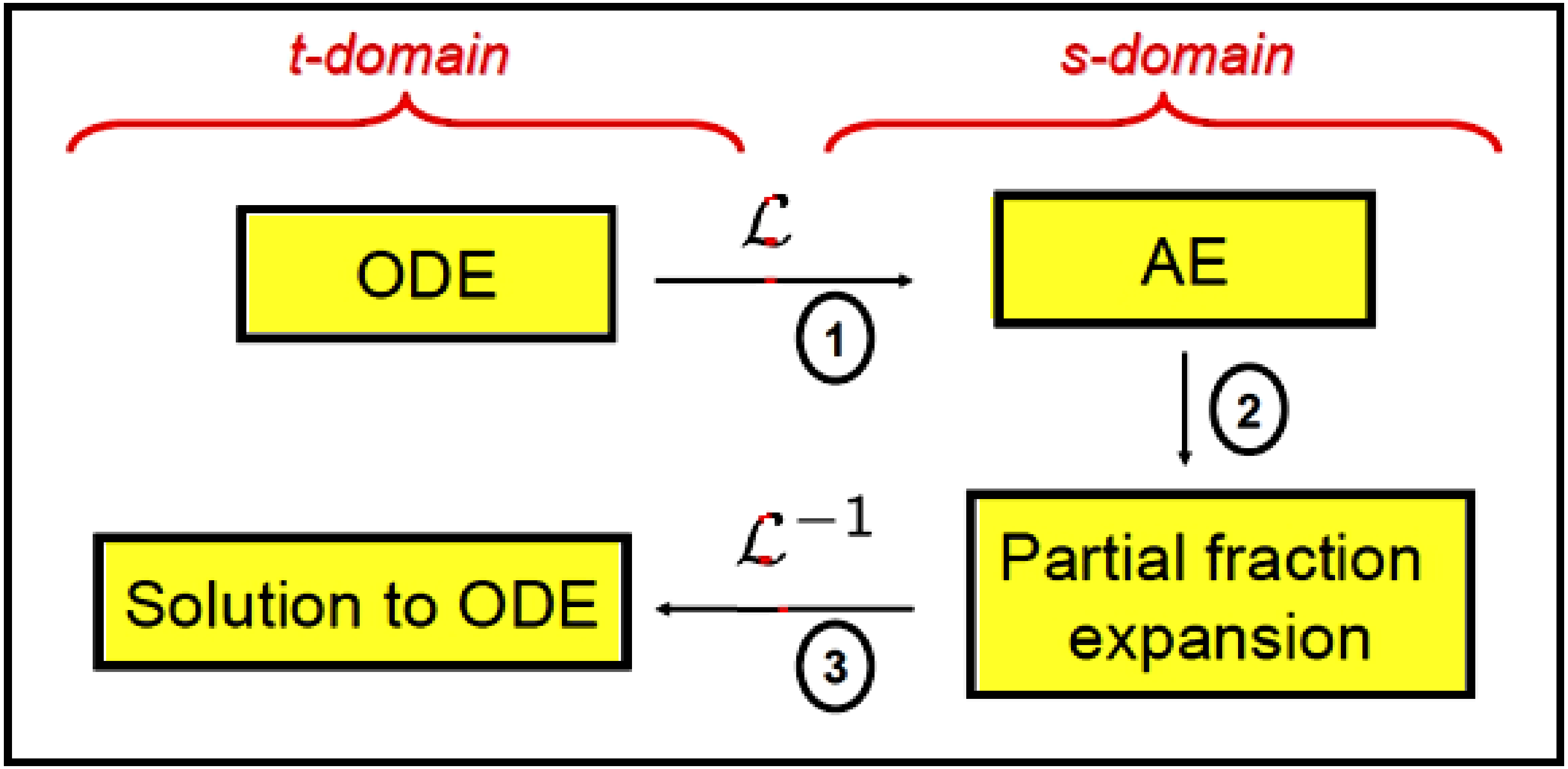
Common Laplace transform pairs

Time function $f(t)$	Laplace transform $\mathscr{L}[f(t)] = F(s)$
unit impulse $\delta(t)$	1
unit step 1	$1/s$
unit ramp t	$1/s^2$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{(s+a)}$
$1 - e^{-at}$	$\frac{a}{s(s+a)}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}(\cos \omega t - \frac{a}{\omega} \sin \omega t)$	$\frac{s}{(s+a)^2+\omega^2}$



An advantage of Laplace transform

- We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).



*** Solution of ODE means SEPARATION of the variables such that :**

t …… in Right Hand Side

y(t) …. In Left Hand Side

*** To find the final value of y(t) then use the final value theorem of laplace**

Example 1 (distinct roots)

ODE with initial conditions (ICs)

$$\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}-2y(t)=0, \quad y(0)=0 \quad , \quad y'(0)=3$$

$$y''(t)+y'(t)-2y(t)=0$$

1. Laplace transform

$$\underbrace{s^2Y(s)-sy(0)-y'(0)}_{\mathcal{L}\left\{y''(t)\right\}}+\underbrace{\{sY(s)-y(0)\}}_{\mathcal{L}\left\{y'(t)\right\}}-\underbrace{2Y(s)}_{\mathcal{L}\left\{2y(t)\right\}}=0$$

$$\Rightarrow \quad Y(s)=\frac{3}{(s-1)(s+2)}$$

Example 1 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{3}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

To obtain A:

Multiply both sides by $(s-1)$ & let s go to $+1$:

$$(s-1)Y(s)|_{s\to+1} = A + (s-1)\frac{B}{(s+2)}\Bigg|_{s\to+1} \Rightarrow A = 1$$

To obtain B:

Multiply both sides by $(s+2)$ & let s go to -2:

$$(s+2)Y(s)|_{s\to-2} = (s+2)\frac{A}{(s-1)}\Bigg|_{s\to-2} + B \Rightarrow B = -1$$

Example 1 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s-1} + \frac{B}{s+2} = \frac{1}{s-1} + \frac{-1}{s+2}$$

$$\Rightarrow \quad y(t) = \left(\underbrace{1}_{A} e^{t} \underbrace{-1}_{B} e^{-2t} \right)$$

If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{3s}{(s-1)(s+2)} = 0$$

Example 2 (distinct roots)

ODE with initial conditions (ICs)

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=5, \quad y(0)=-1, \quad y'(0)=2$$

$$y''(t)+3y'(t)+2y(t)=5$$

1. Laplace transform

$$\underbrace{s^2Y(s)-sy(0)-y'(0)}_{\mathcal{L}\{y''(t)\}}+3\underbrace{\{sY(s)-y(0)\}}_{\mathcal{L}\{y'(t)\}}+2Y(s)=\frac{5}{s}$$

$$\Rightarrow \quad Y(s)=\frac{-s^2-s+5}{s(s+1)(s+2)}$$ ← *distinct roots*

Example 2 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Multiply both sides by s & let s go to zero:

$$sY(s)|_{s\to 0} = A + s\frac{B}{s+1}\bigg|_{s\to 0} + s\frac{C}{s+2}\bigg|_{s\to 0} \Rightarrow A = sY(s)|_{s\to 0} = \frac{5}{2}$$

Similarly,

$$B = (s+1)Y(s)|_{s\to -1} = \cdots = -5$$
$$C = (s+2)Y(s)|_{s\to -2} = \cdots = \frac{3}{2}$$

Example 2 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{\frac{5}{2}}{s} + \frac{-5}{s+1} + \frac{\frac{3}{2}}{s+2}$$

$$\Rightarrow \quad y(t) = \left(\underbrace{\frac{5}{2}}_{A} + \underbrace{(-5)}_{B} e^{-t} + \underbrace{\frac{3}{2}}_{C} e^{-2t} \right)$$

If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{-s^2 - s + 5}{(s+1)(s+2)} = \frac{5}{2}$$

Example 3 (repeated roots)

ODE with initial conditions (ICs)

$$\frac{d^3y(t)}{dt^3}+5\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+4y(t)=2\delta(t), \quad y(0)=y'(0)=y''(0)=0$$

$$y'''(t)+5\,y''(t)+8\,y'(t)+4y(t)=2\delta(t)$$

1. Laplace transform

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0) \longleftarrow \mathcal{L}\left\{y'''(t)\right\}$$
$$+5\left\{s^2Y(s)-sy(0)-y'(0)\right\} \longleftarrow \mathcal{L}\left\{5\,y''(t)\right\}$$
$$+8\left\{sY(s)-y(0)\right\} \longleftarrow \mathcal{L}\left\{8\,y'(t)\right\}$$
$$+4Y(s) \longleftarrow \mathcal{L}\left\{4\,y(t)\right\}$$

$$\Rightarrow \quad Y(s)=\frac{2}{(s+1)(s+2)^2}$$

Repeated roots

Example 3 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

To obtain A:

$$(s+1)Y(s)\big|_{s\to -1} = A + (s+1)\frac{B}{(s+2)^2}\bigg|_{s\to -1} + (s+1)\frac{C}{s+2}\bigg|_{s\to -1} \Rightarrow A = 2$$

To obtain B:

$$(s+2)^2 Y(s)\big|_{s\to -2} = (s+2)^2\frac{A}{s+1}\bigg|_{s\to -2} + B + (s+2)C\big|_{s\to -2} \Rightarrow B = -2$$

Example 3 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

Put S=0 in both sides and substitute about values of A & B … then C = -2

$$C = -2$$

Example 3 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+2)^2} + \frac{C}{s+2} = \frac{2}{s+1} + \frac{-2}{(s+2)^2} + \frac{-2}{s+2}$$

$$\Rightarrow \quad y(t) = \left(\underbrace{2}_{A} e^{-t} + \underbrace{(-2)}_{B} te^{-2t} + \underbrace{-2}_{C} e^{-2t} \right)$$

If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{2s}{(s+1)(s+2)^2} = 0$$